

GGG-GROUPS: ORDER OF CONGRUENCE QUOTIENTS AND HAUSDORFF DIMENSION

GUSTAVO A. FERNÁNDEZ-ALCOBER AND AMAIA ZUGADI-REIZABAL

ABSTRACT. If G is a GGS-group defined over a p -adic tree, where p is an odd prime, we calculate the order of the congruence quotients $G_n = G/\text{Stab}_G(n)$ for every n . If G is defined by the vector $\mathbf{e} = (e_1, \dots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, the determination of the order of G_n is split into three cases, according as \mathbf{e} is non-symmetric, non-constant symmetric, or constant. The formulas that we obtain only depend on p , n , and the rank of the circulant matrix whose first row is \mathbf{e} . As a consequence of these formulas, we also obtain the Hausdorff dimension of the closures of all GGS-groups over the p -adic tree.

1. INTRODUCTION

Subgroups of the group of automorphisms of a regular rooted tree have turned out to be a source of many interesting examples in group theory. Particular attention has been given to the so-called Grigorchuk groups and to the Gupta-Sidki group, introduced in [10] and [11], respectively. The second of the Grigorchuk groups and the Gupta-Sidki group are particular instances of the family of *GGG-groups* (GGG after Grigorchuk, Gupta, and Sidki, a term coined by Gilbert Baumslag), to which this paper is devoted. We work over the p -adic tree, where p is an odd prime, and we determine the order of all congruence quotients of GGS-groups; these are the automorphism groups induced by GGS-groups on the finite trees which are obtained by truncating the p -adic tree at every level. As a consequence, we also obtain the Hausdorff dimension of the closures of GGS-groups.

Before defining GGS-groups and stating our main results, it is convenient to recall some concepts from the theory of automorphisms of rooted trees. If $m \geq 2$ is an integer and $X = \{1, \dots, m\}$, the m -adic tree \mathcal{T} is the tree whose set of vertices is the free monoid X^* , where a word u is a descendant of v if $u = vx$ for some $x \in X$. If we consider only words of length $\leq n$, then we have a finite tree \mathcal{T}_n , which we refer to as *the tree \mathcal{T} truncated at level n* . The group $\text{Aut } \mathcal{T}$ of all automorphisms of \mathcal{T} is a profinite group with respect to the topology induced by the filtration of the level stabilizers $\text{Stab}(n)$, and we have $\text{Aut } \mathcal{T} \cong \varprojlim_n \text{Aut } \mathcal{T}_n$. The stabilizer $\text{Stab}(n)$ of the n th level of \mathcal{T} is the normal subgroup of $\text{Aut } \mathcal{T}$ consisting of all automorphisms leaving fixed all words of length n (and, consequently, also all vertices of \mathcal{T}_n). These stabilizers can be considered as natural congruence subgroups for $\text{Aut } \mathcal{T}$. If G is a subgroup of $\text{Aut } \mathcal{T}$ and we put $\text{Stab}_G(n) = \text{Stab}(n) \cap G$, then we

Supported by the Spanish Government, grant MTM2008-06680-C02-02, partly with FEDER funds, and by the Basque Government, grant IT-460-10. The second author is also supported by grant BFI07.95 of the Basque Government.

refer to the quotient $G_n = G/\text{Stab}_G(n)$ as the *n*th congruence quotient of G . Since the kernel of the action of G on \mathcal{T}_n is $\text{Stab}_G(n)$, it follows that G_n can be naturally seen as a subgroup of $\text{Aut } \mathcal{T}_n$.

If an automorphism g fixes a vertex u , then the restriction of g to the subtree hanging from u induces an automorphism g_u of \mathcal{T} . In particular, if $g \in \text{Stab}(1)$ then g_i is defined for every $i = 1, \dots, m$, and we can consider the map

$$\begin{aligned} \psi : \text{Stab}(1) &\longrightarrow \text{Aut } \mathcal{T} \times \cdots \times \text{Aut } \mathcal{T} \\ g &\longmapsto (g_1, \dots, g_m). \end{aligned}$$

Clearly, ψ is a group isomorphism.

On the other hand, any $g \in \text{Aut } \mathcal{T}$ can be completely determined by describing how g sends the descendants of every vertex u to the descendants of $g(u)$. This can be done by indicating, for every $x \in X$, the element $\alpha(x) \in X$ such that $g(ux) = g(u)\alpha(x)$. Then α is a permutation of X , which we call the *label* of g at u , and we denote by $g_{(u)}$. The set of all labels of g constitutes the *portrait* of g . Thus g is determined by its portrait. We have the following rules for labels under composition and inversion:

$$(1) \quad (fg)_{(u)} = f_{(u)}g_{(f(u))} \quad \text{and} \quad (f^{-1})_{(u)} = (f_{(f^{-1}(u))})^{-1}.$$

An important automorphism of \mathcal{T} is the automorphism that permutes the m subtrees hanging from the root rigidly according to the permutation $(1 \ 2 \ \dots \ m)$. This is called a *rooted automorphism* and will be denoted by the letter a . Since a has order m , it makes sense to write a^k for $k \in \mathbb{Z}/m\mathbb{Z}$. Now, given a non-zero vector $\mathbf{e} = (e_1, \dots, e_{m-1}) \in (\mathbb{Z}/m\mathbb{Z})^{m-1}$, we can define recursively an automorphism b of \mathcal{T} via

$$\psi(b) = (a^{e_1}, \dots, a^{e_{m-1}}, b).$$

We say that the subgroup $G = \langle a, b \rangle$ of $\text{Aut } \mathcal{T}$ is the *GGs-group* corresponding to the *defining vector* \mathbf{e} . If $m = 2$ then there is only one GGS-group, which is isomorphic to D_∞ , the infinite dihedral group. The second Grigorchuk group is obtained by choosing $m = 4$ and $\mathbf{e} = (1, 0, 1)$, and the Gupta-Sidki group arises for m equal to an odd prime and $\mathbf{e} = (1, -1, 0, \dots, 0)$. The groups corresponding to $\mathbf{e} = (1, 0, \dots, 0)$ and arbitrary m have also deserved special attention. In the case $m = 3$, this group was introduced by Fabrykowski and Gupta in [8]. As a reference for GGS-groups, the reader can consult Section 2.3 of the monograph [5] by Bartholdi, Grigorchuk, and Šunić, the habilitation thesis [14] of Rozhkov, or the papers [18] by Vovkivsky and [12, 13] by Pervova.

Little is known about the orders of the congruence quotients G_n when G is a GGS-group. In the case that $\mathbf{e} = (1, 0, \dots, 0)$ and $m = p$ is a prime, Šunić found in [17] that, for every $n \geq 2$,

$$\log_p |G_n| = \begin{cases} p^{n-1} + 1, & \text{if } p \text{ is odd,} \\ 2^{n-2} + 2, & \text{if } p = 2. \end{cases}$$

Hence we may always assume that $m \geq 3$, as far as the problem of determining $|G_n|$ is concerned. To the best of our knowledge, the only other cases

in which the order of G_n has been determined for every n correspond to $m = 3$. For the Gupta-Sidki group, Sidki himself (see [15]) proved that

$$\log_3 |G_n| = 2 \cdot 3^{n-2} + 1, \quad \text{for every } n \geq 2.$$

On the other hand, for $\mathbf{e} = (1, 1)$, Bartholdi and Grigorchuk showed in [4] that

$$\log_3 |G_n| = \frac{3^n + 2n + 3}{4}, \quad \text{for every } n \geq 2.$$

From now onwards, we assume that m is equal to an odd prime p , and so \mathcal{T} stands for the p -adic tree. The first of our main results is the determination of the order of G_n for *all* GGS-groups under this assumption. Before giving the statement of the theorem, we introduce some notation. Given a vector $\mathbf{a} = (a_1, \dots, a_n)$, we write $C(\mathbf{a})$ to denote the circulant matrix generated by \mathbf{a} , i.e. the matrix of size $n \times n$ whose first row is \mathbf{a} , and every other row is obtained from the previous one by applying a shift of length one to the right. In other words, the entries of $C(\mathbf{a})$ are $c_{ij} = a_{j-i+1}$, where a_k is defined for every integer k by reducing k modulo n to a number between 1 and n . If \mathbf{e} is the defining vector of a GGS-group, then we write $C(\mathbf{e}, 0)$ for the circulant matrix $C(e_1, \dots, e_{p-1}, 0)$ over \mathbb{F}_p . We say that \mathbf{e} is *symmetric* if $e_i = e_{p-i}$ for all $i = 1, \dots, p-1$.

Theorem A. *Let G be a GGS-group over the p -adic tree, where p is an odd prime, and let \mathbf{e} be the defining vector of G . Then, for every $n \geq 2$, we have*

$$\log_p |G_n| = tp^{n-2} + 1 - \delta \frac{p^{n-2} - 1}{p-1} - \varepsilon \frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

where t is the rank of the circulant matrix $C(\mathbf{e}, 0)$,

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

If $\sigma = (1 \ 2 \ \dots \ p)$, then the automorphisms whose portrait consists only of powers of σ form a Sylow pro- p subgroup of $\text{Aut } \mathcal{T}$, which we denote by Γ . Observe that, under the assumption $m = p$ that we have made, all GGS-groups are subgroups of Γ . According to Theorem 1 of [18], the requirement that \mathbf{e} is non-zero implies that GGS-groups are infinite if $m = p$. Since they are countable groups, they cannot be closed in the pro- p group Γ . Our second main result is related to the Hausdorff dimension of the closures of GGS-groups.

The determination of the Hausdorff dimension of closed subgroups of Γ has received special attention in the last few years (see [2, 9, 16, 17]). The most natural choice is to calculate the Hausdorff dimension with respect to the metric induced by the filtration of Γ given by the level stabilizers $\text{Stab}_\Gamma(n)$. In this case, it follows from a result of Abercrombie [1], and Barnea and Shalev [3], that the Hausdorff dimension of the closure \overline{G} of a subgroup G of Γ is given by the following formula:

$$(2) \quad \dim_\Gamma \overline{G} = \liminf_{n \rightarrow \infty} \frac{\log_p |G_n|}{\log_p |\Gamma_n|} = (p-1) \liminf_{n \rightarrow \infty} \frac{\log_p |G_n|}{p^n}.$$

As an immediate consequence of Theorem A, we get the Hausdorff dimension of the closure of any GGS-group.

Theorem B. *Let G be a GGS-group over the p -adic tree, where p is an odd prime, and let \mathbf{e} be the defining vector of G . Then*

$$\dim_{\Gamma} \overline{G} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

where t is the rank of the circulant matrix $C(\mathbf{e}, 0)$,

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

Our proof of Theorem A relies on finding some kind of branch structure inside a GGS-group G . In particular, if \mathbf{e} is not constant, we show that G is regular branch (see Section 3 for the definition). This result had been previously proved by Pervova and Rozhkov for *periodic* GGS-groups. On the other hand, it is worth mentioning that the theory of p -groups of maximal class plays also a crucial role in the proof of Theorem A, particularly in the case that \mathbf{e} is constant.

Notation. The i th row and j th column of a matrix C are denoted by C_i and C^j , respectively.

2. GENERAL PROPERTIES OF GGS-GROUPS

Throughout the paper, a and b denote the canonical generators of a GGS-group G , and $b_i = b^{a^i}$ for every integer i . Note that $b_i = b_j$ if $i \equiv j \pmod{p}$. The images of the elements b_i under the map ψ of the introduction can be easily described:

$$(3) \quad \begin{aligned} \psi(b_0) &= (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b), \\ \psi(b_1) &= (b, a^{e_1}, \dots, a^{e_{p-2}}, a^{e_{p-1}}), \\ &\vdots \\ \psi(b_{p-1}) &= (a^{e_2}, a^{e_3}, \dots, b, a^{e_1}). \end{aligned}$$

We begin with some easy facts about GGS-groups.

Theorem 2.1. *If $G = \langle a, b \rangle$ is a GGS-group, then:*

- (i) $\text{Stab}_G(1) = \langle b \rangle^G = \langle b_0, \dots, b_{p-1} \rangle$ and $G = \langle a \rangle \rtimes \text{Stab}_G(1)$.
- (ii) $\text{Stab}_G(2) \leq G' \leq \text{Stab}_G(1)$.
- (iii) $|G : G'| = p^2$ and $|G : \gamma_3(G)| = p^3$.

Proof. One can easily check the equalities in part (i). Thus $G/\text{Stab}_G(1)$ is cyclic and $G' \leq \text{Stab}_G(1)$.

The quotient $G/G' = \langle aG', bG' \rangle$ is elementary abelian of order at most p^2 . It follows that $G'/\gamma_3(G) = \langle [a, b]\gamma_3(G) \rangle$ has order at most p . If $G' = \gamma_3(G)$ then $\gamma_i(G) = G'$ for every $i \geq 3$. On the other hand, since G is residually a finite p -group, the intersection of all the $\gamma_i(G)$ is trivial. Consequently $G' = 1$, which is a contradiction, since $b^a \neq b$ by (3). We conclude that

$|G' : \gamma_3(G)| = p$. Now, if $|G : G'| \leq p$ then G/G' is cyclic, and $G' = \gamma_3(G)$. Hence we necessarily have $|G : G'| = p^2$, and (iii) follows.

It only remains to prove that $N = \text{Stab}_G(2)$ is contained in G' . Since $|G : G'| = p^2$, it suffices to prove that $|G/N : (G/N)'| = p^2$. If $|G/N : (G/N)'| \leq p$ then G/N , being a finite p -group, must be cyclic. This is a contradiction, since $\langle aN \rangle$ and $\langle bN \rangle$ are two different subgroups of order p in G/N . (Note that $\langle bN \rangle$ is contained in $\text{Stab}_G(1)/N$ while $\langle aN \rangle$ is not.) \square

Now if $g \in \text{Stab}_G(1)$, it readily follows from (3) and the previous theorem that $g_i \in G$ for all $i = 1, \dots, p$. Thus the image of $\text{Stab}_G(1)$ under ψ is actually contained in $G \times \cdots \times G$, and so

$$(4) \quad \psi(\text{Stab}_G(k)) \subseteq \text{Stab}_G(k-1) \times \cdots \times \text{Stab}_G(k-1)$$

for all $k \geq 1$. Another important property of the map ψ is the following.

Proposition 2.2. *If G is a GGS-group, then the composition of ψ with the projection on any component is surjective from $\text{Stab}_G(1)$ onto G .*

Proof. Let us fix a position $i \in \{1, \dots, p\}$, and let $j \in \{1, \dots, p-1\}$ be such that $e_j \neq 0$. It follows from (3) that $\psi(b_{i-j})$ and $\psi(b_i)$ have the entries a^{e_j} and b in the i th component. Since $G = \langle a, b \rangle = \langle a^{e_j}, b \rangle$, the result follows. \square

For every positive integer n , we can define an isomorphism ψ_n from the stabilizer of the first level in $\text{Aut } \mathcal{T}_n$ to the direct product $\text{Aut } \mathcal{T}_{n-1} \times \cdots \times \text{Aut } \mathcal{T}_{n-1}$, in the same way as ψ is defined. Since G_n can be seen as a subgroup of $\text{Aut } \mathcal{T}_n$, we can consider the restriction of ψ_n to $\text{Stab}_{G_n}(1)$. It follows from (4) that

$$\psi_n(\text{Stab}_{G_n}(k)) \subseteq \text{Stab}_{G_{n-1}}(k-1) \times \cdots \times \text{Stab}_{G_{n-1}}(k-1).$$

Obviously, G_1 is of order p , generated by the image \bar{a} of a . Next we deal with G_2 . Let us write \tilde{g} for the image of an element $g \in G$ in G_2 . Since $G_2 = \langle \bar{a} \rangle \rtimes \text{Stab}_{G_2}(1)$, it suffices to understand $\text{Stab}_{G_2}(1) = \langle \tilde{b}_0, \dots, \tilde{b}_{p-1} \rangle$. Observe that ψ_2 sends $\text{Stab}_{G_2}(1)$ into $G_1 \times \cdots \times G_1$, which can be identified with \mathbb{F}_p^p under the linear map

$$(\bar{a}^{i_1}, \dots, \bar{a}^{i_p}) \mapsto (i_1, \dots, i_p).$$

This allows us to consider $\text{Stab}_{G_2}(1)$ as a vector space over \mathbb{F}_p .

Before analyzing G_2 in the next theorem, we need the following lemma (see Exercise 4 in Section 1 of the book [6]) about finite p -groups of maximal class, which will be also used at some other places in the paper.

Lemma 2.3. *Let P be a finite p -group such that $|P : P'| = p^2$. If P has an abelian maximal subgroup A , then P is a group of maximal class. Furthermore, if $g_0 \in P \setminus A$, then:*

- (i) *If $a \in A \setminus \gamma_2(P)$, then $\gamma_2(P)/\gamma_3(P)$ is generated by the image of $[a, g_0]$.*
- (ii) *If $i \geq 2$ and $a \in \gamma_i(P) \setminus \gamma_{i+1}(P)$, then $\gamma_{i+1}(P)/\gamma_{i+2}(P)$ is generated by the image of $[a, g_0]$.*

Theorem 2.4. *Let G be a GGS-group with defining vector \mathbf{e} , and put $C = C(\mathbf{e}, 0)$. Then:*

- (i) *The dimension of $\text{Stab}_{G_2}(1)$ coincides with the rank t of C .*
- (ii) *G_2 is a p -group of maximal class of order p^{t+1} .*

Proof. (i) If $\tilde{g} \in \text{Stab}_{G_2}(1)$ and $\psi_2(\tilde{g}) = (\bar{a}^{i_1}, \dots, \bar{a}^{i_p})$, where we consider the exponents i_1, \dots, i_p as elements of \mathbb{F}_p , we define

$$\Psi_2(\tilde{g}) = (i_1, \dots, i_p) \in \mathbb{F}_p^p.$$

Observe that Ψ_2 is injective.

By (3),

$$\Psi_2(\tilde{b}_0) = (e_1, e_2, \dots, e_{p-1}, 0) = (\mathbf{e}, 0)$$

coincides with the first row of C . Since the components of the rest of the b_i are obtained by permuting cyclically those of b_0 , and since $C = C(\mathbf{e}, 0)$, it follows that $\Psi_2(\tilde{b}_i)$ is the $(i+1)$ st row of C . Thus the dimension of $\text{Stab}_{G_2}(1)$ coincides with the dimension of the subspace of \mathbb{F}_p^p generated by the rows of C , i.e. with the rank t of the matrix C .

(ii) We have

$$|G_2| = |G_2 : \text{Stab}_{G_2}(1)| |\text{Stab}_{G_2}(1)| = p \cdot p^t = p^{t+1}.$$

On the other hand, it follows from (ii) and (iii) of Theorem 2.1 that $|G_2 : G'_2| = p^2$. Since $\text{Stab}_{G_2}(1)$ is an abelian maximal subgroup of G_2 , we conclude from Lemma 2.3 that G_2 is a p -group of maximal class. \square

As a consequence, we can improve part (ii) of Theorem 2.1.

Corollary 2.5. *If G is a GGS-group, then $\text{Stab}_G(2) \leq \gamma_3(G)$.*

Proof. Since the defining vector \mathbf{e} of G is different from $(0, \dots, 0)$, it is clear that the rank t of the matrix $C(\mathbf{e}, 0)$ is at least 2. It follows from the previous theorem that $G_2 = G / \text{Stab}_G(2)$ is a p -group of maximal class of order greater than or equal to p^3 . Thus $|G_2 : \gamma_3(G_2)| = p^3 = |G : \gamma_3(G)|$, and consequently $\text{Stab}_G(2)$ is contained in $\gamma_3(G)$. \square

We have seen in Theorem 2.1 that $G' \leq \text{Stab}_G(1)$. Next we want to characterize which elements of $\text{Stab}_G(1)$ belong to G' . This goal will be achieved in Theorem 2.11. If $g \in \text{Stab}_G(1) = \langle b_0, \dots, b_{p-1} \rangle$, then we can write g as a word in b_0, \dots, b_{p-1} , i.e. we can write $g = \omega(b_0, \dots, b_{p-1})$, where $\omega = \omega(x_0, \dots, x_{p-1})$ is a group word in the p variables x_0, \dots, x_{p-1} .

Definition 2.6. Let ω be a group word in the variables x_0, \dots, x_{p-1} , where p is a prime. Then:

- (i) The *partial p -weight* of ω with respect to a variable x_i , with $0 \leq i \leq p-1$, is the sum of the exponents of x_i in the expression for ω , considered as an element of \mathbb{F}_p .
- (ii) The *total p -weight* of ω is the sum of all its partial p -weights.

It is not difficult to give examples showing that the representation of an element $g \in \text{Stab}_G(1)$ as a word in b_0, \dots, b_{p-1} is not unique. Our first step towards the proof of Theorem 2.11 will be to see that, however, the partial and total p -weights are the same for all word representations. For this purpose, we need the following lemma.

Lemma 2.7. *Let p be a prime, and let $(a_0, \dots, a_{p-1}) \in \mathbb{F}_p^p$ be a non-zero vector. If $C = C(a_0, \dots, a_{p-1})$, then:*

- (i) $\text{rk } C = p - m$, where m is the multiplicity of 1 as a root of the polynomial $a(X) = a_0 + a_1X + \dots + a_{p-1}X^{p-1}$. As a consequence, we have $\text{rk } C < p$ if and only if $\sum_{i=0}^{p-1} a_i = 0$.
- (ii) If $\mathbf{1}$ represents the column vector of length p with all entries equal to 1, then

$$\text{rk } C = \text{rk } (C \mid \mathbf{1}).$$

Proof. If we consider the quotient ring $V = \mathbb{F}_p[X]/(X^p - 1)$ as an \mathbb{F}_p -vector space, then both

$$\mathcal{B} = \{\overline{1}, \overline{X}, \dots, \overline{X^{p-1}}\}$$

and

$$\mathcal{B}' = \{\overline{1}, \overline{X-1}, \dots, \overline{(X-1)^{p-1}}\}$$

are bases of V . Multiplication by $\overline{a(X)}$ defines a linear map $\varphi : V \rightarrow V$, and the matrix of φ with respect to \mathcal{B} is C (we construct the matrix by rows). Thus $\text{rk } C = \text{rk } \varphi$.

On the other hand, we can write $a(X) = (X-1)^m b(X)$, with $b(X) \in \mathbb{F}_p[X]$ and $b(1) \neq 0$. Let $b(X) = b_0 + b_1(X-1) + \dots + b_{k-1}(X-1)^{k-1}$, where $k = p - m$ and $b_0 \neq 0$. Then the matrix of φ with respect to \mathcal{B}' is the block matrix

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} b_0 & b_1 & \dots & b_{k-2} & b_{k-1} \\ 0 & b_0 & \dots & b_{k-3} & b_{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_0 \end{pmatrix},$$

since $\overline{(X-1)^i} = \overline{0}$ in V for all $i \geq p$. Thus $\text{rk } \varphi = k$, and (i) follows.

Let us now prove (ii). We first prove that

$$(5) \quad \text{rk } C = \text{rk } \begin{pmatrix} C \\ 1 \dots 1 \end{pmatrix}.$$

Since C is the matrix of φ with respect to \mathcal{B} constructed by rows, it is clear that (5) is equivalent to $1 + X + \dots + X^{p-1}$ lying in the image of φ . Note that, since we are working with coefficients in \mathbb{F}_p , we have

$$1 + X + \dots + X^{p-1} = (X-1)^{p-1}.$$

Since

$$\varphi(\overline{(X-1)^{k-1}}) = \overline{b_0(X-1)^{p-1}},$$

and $b_0 \neq 0$, it follows that $\overline{(X-1)^{p-1}} \in \text{im } \varphi$, as desired.

Now, since the transpose ${}^t C$ of C is also a circulant matrix, we can apply (5) to ${}^t C$ and get

$$\text{rk } C = \text{rk } {}^t C = \text{rk } \begin{pmatrix} {}^t C \\ 1 \dots 1 \end{pmatrix} = \text{rk } {}^t (C \mid \mathbf{1}) = \text{rk } (C \mid \mathbf{1}).$$

□

Let $g = \omega(b_0, \dots, b_{p-1})$ be an arbitrary element of $\text{Stab}_G(1)$, and suppose that the partial p -weight of ω with respect to x_i is r_i , for $i = 0, \dots, p-1$. It follows from (3) that

$$(6) \quad \psi(g) = (a^{m_1}\omega_1(b_0, \dots, b_{p-1}), \dots, a^{m_p}\omega_p(b_0, \dots, b_{p-1})),$$

where each ω_i is a word of total p -weight r_i (and where r_p is to be understood as r_0), and

$$(7) \quad m_i = (r_0 \ r_1 \ \dots \ r_{p-1})C^i.$$

Theorem 2.8. *Let G be a GGS-group, and let $g \in \text{Stab}_G(1)$. Then the partial and total p -weights are the same for all representations of g as a word in b_0, \dots, b_{p-1} .*

Proof. It suffices to see that, if ω is a word such that $\omega(b_0, \dots, b_{p-1}) = 1$, then the total p -weight of ω is 0, and the partial p -weight r_i of ω with respect to x_i is equal to 0, for every $i = 0, \dots, p-1$. Obviously, the second assertion implies the first one, but the proof will go the other way around.

As in (6), we write

$$(8) \quad \psi(\omega(b_0, \dots, b_{p-1})) = (a^{m_1}\omega_1(b_0, \dots, b_{p-1}), \dots, a^{m_p}\omega_p(b_0, \dots, b_{p-1})).$$

Since this element is equal to 1, it follows that $m_i = 0$ for $i = 1, \dots, p$. According to (7), this means that

$$(r_0 \ r_1 \ \dots \ r_{p-1})C = (0 \ 0 \ \dots \ 0).$$

Now, since $\text{rk } C = \text{rk}(C \mid \mathbf{1})$ by Lemma 2.7, we also have $(r_0 \ r_1 \ \dots \ r_{p-1})\mathbf{1} = 0$, that is,

$$r_0 + r_1 + \dots + r_{p-1} = 0.$$

This proves that the total p -weight of ω is 0.

Now we return to (8). Since $\omega(b_0, \dots, b_{p-1}) = 1$ by hypothesis, then we also have $\omega_i(b_0, \dots, b_{p-1}) = 1$ for all $i = 1, \dots, p$. Now, since the total p -weight of ω_i is r_i , it follows from the previous paragraph that $r_i = 0$. \square

The independence of the partial and total p -weights from the word representation allows us to give the following definition.

Definition 2.9. Let G be a GGS-group, and let $g \in \text{Stab}_G(1)$. We define the *partial weight* of g with respect to b_i , and the *total weight* of g , as the corresponding p -weights for any word ω representing g .

We prefer to speak simply about weights instead of p -weights in the case of an element $g \in \text{Stab}_G(1)$, since all elements b_i (with respect to which the weights are considered) have order p . Now the following result is clear.

Theorem 2.10. *Let G be a GGS-group. Then the maps from $\text{Stab}_G(1)$ to \mathbb{F}_p sending every $g \in \text{Stab}_G(1)$ to its partial weight with respect to one of the b_i or to its total weight are well-defined homomorphisms.*

Theorem 2.11. *Let G be a GGS-group. Then the derived subgroup G' consists of all the elements of $\text{Stab}_G(1)$ whose total weight is equal to 0.*

Proof. The map ϑ sending each element of $\text{Stab}_G(1)$ to its total weight is a homomorphism onto the abelian group \mathbb{F}_p , and consequently $G' \leq \ker \vartheta$. Since $|G : G'| = p^2$ and $|G : \text{Stab}_G(1)| = |\text{Stab}_G(1) : \ker \vartheta| = p$, the equality follows. \square

Definition 2.12. Let G be a GGS-group. If $g \in \text{Stab}_G(1)$ has partial weight r_i with respect to b_i for $i = 0, \dots, p-1$, we say that $(r_0, \dots, r_{p-1}) \in \mathbb{F}_p^p$ is the *weight vector* of g .

As we next see, we can analyze the subgroups $\text{Stab}_G(2)$ and $\text{Stab}_G(3)$ by using the weight vector.

Theorem 2.13. Let G be a GGS-group with defining vector \mathbf{e} , and put $C = C(\mathbf{e}, 0)$. If the weight vector of $g \in \text{Stab}_G(1)$ is (r_0, \dots, r_{p-1}) , then:

- (i) We have $g \in \text{Stab}_G(2)$ if and only if $(r_0 \dots r_{p-1})C = (0 \dots 0)$.
- (ii) If $g \in \text{Stab}_G(3)$ then $(r_0, \dots, r_{p-1}) = (0, \dots, 0)$.

Proof. (i) If we write $\psi(g)$ as in (6), then $g \in \text{Stab}_G(2)$ if and only if $m_i = 0$ in \mathbb{F}_p for every $i = 1, \dots, p$. Now, by (7), this is equivalent to the condition $(r_0 \dots r_{p-1})C = (0 \dots 0)$.

(ii) Again we use the expression in (6). If $g \in \text{Stab}_G(3)$ then $\omega_i(b_0, \dots, b_{p-1}) \in \text{Stab}_G(2)$ for all $i = 1, \dots, p$. As mentioned above, $\omega_i(b_0, \dots, b_{p-1})$ is an element of total weight r_i . Let (s_0, \dots, s_{p-1}) be the weight vector of this element, so that $r_i = s_0 + \dots + s_{p-1}$. Then, by (i), we have $(s_0 \dots s_{p-1})C = (0 \dots 0)$. Since $\text{rk } C = \text{rk}(C \mid \mathbf{1})$ by Lemma 2.7, it follows that $r_i = s_0 + \dots + s_{p-1} = 0$, as desired. \square

One may wonder whether the converse holds in (ii) of the previous theorem, i.e. if the weight vector of an element is $(0, \dots, 0)$, does it lie in $\text{Stab}_G(3)$? We make things clearer in the following theorem.

Theorem 2.14. Let G be a GGS-group. Then $\text{Stab}_G(1)'$ consists of all elements of $\text{Stab}_G(1)$ whose weight vector is $(0, \dots, 0)$. Furthermore, we have $|G : \text{Stab}_G(1)'| = p^{p+1}$.

Proof. The map ρ which sends every element of $\text{Stab}_G(1)$ to its weight vector is a homomorphism onto \mathbb{F}_p^p . Thus $|\text{Stab}_G(1) : \ker \rho| = p^p$. Since \mathbb{F}_p^p is abelian, it follows that $\text{Stab}_G(1)' \leq \ker \rho$. On the other hand, since $\text{Stab}_G(1) = \langle b_0, \dots, b_{p-1} \rangle$ and every b_i has order p , we have $|\text{Stab}_G(1) : \text{Stab}_G(1)'| \leq p^p$. Hence $\ker \rho = \text{Stab}_G(1)'$ and $|\text{Stab}_G(1) : \text{Stab}_G(1)'| = p^p$. Since $|G : \text{Stab}_G(1)| = p$, we are done. \square

In particular, we have $\text{Stab}_G(3) \leq \text{Stab}_G(1)'$. Once we prove Theorem A, it will follow that $|G : \text{Stab}_G(3)| = p^{tp+1-\delta}$, where t is the rank of $C(\mathbf{e}, 0)$ and δ is 1 or 0, according as \mathbf{e} is symmetric or not. Since t is always at least 2, we have $|G : \text{Stab}_G(3)| > p^{p+1}$ in every case. Hence $\text{Stab}_G(3)$ is always a proper subgroup of $\text{Stab}_G(1)'$, and the converse of (ii) in Theorem 2.13 never holds.

Next we prove a result which will allow us to reduce, for the calculation of the order of congruence quotients and of the Hausdorff dimension, to the case of GGS-groups with defining vectors of the form $\mathbf{e} = (1, e_2, \dots, e_{p-1})$. We need the following lemma.

Lemma 2.15. Let p be a prime, and let $\sigma = (1 \ 2 \ \dots \ p)$. Assume that $\alpha \in S_p$ satisfies the following two conditions:

- (i) α normalizes the subgroup $\langle \sigma \rangle$.
- (ii) $\alpha(p) = p$.

Then, for every $i = 1, \dots, p-1$, if $\alpha(i) = j$ we have $\alpha(p-i) = p-j$.

Proof. If we think of S_p as the set of permutations of the field \mathbb{F}_p , then σ corresponds to the map $\ell \mapsto \ell+1$, and the normalizer of $\langle \sigma \rangle$ in S_p corresponds to the affine group over \mathbb{F}_p (see Lemma 14.1.2 of [7]). Thus $\alpha(\ell) = a\ell + b$ for some $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$. Since $\alpha(p) = p$, it follows that $b = 0$, and so $\alpha(\ell) = a\ell$ for every $\ell \in \mathbb{F}_p$. Hence α is a linear map and, as a consequence,

$$\alpha(p-i) = \alpha(-i) = -\alpha(i) = -j = p-j.$$

□

We say that an automorphism f of \mathcal{T} has *constant portrait* if f has the same label at all vertices of \mathcal{T} . By formula (1) for the labels of a composition, the set of all automorphisms of constant portrait is a subgroup of $\text{Aut } \mathcal{T}$.

Theorem 2.16. *Let G be a GGS-group with defining vector $\mathbf{e} = (e_1, \dots, e_{p-1})$, and assume that $e_k \neq 0$. Then there exists $f \in \text{Aut } \mathcal{T}$ of constant portrait such that $L = G^f$ is a GGS-group whose defining vector $\mathbf{e}' = (e'_1, \dots, e'_{p-1})$ satisfies:*

- (i) \mathbf{e}' is a permutation of the vector \mathbf{e}/e_k , that is, there exists $\alpha \in S_{p-1}$ such that $e'_i = e_{\alpha(i)}/e_k$ for all $i = 1, \dots, p-1$.
- (ii) $\alpha(1) = k$, and so $e'_1 = 1$.
- (iii) If $\alpha(i) = j$ then $\alpha(p-i) = p-j$. In other words, two values which are placed in symmetric positions of \mathbf{e} are moved (after division by e_k) to symmetric positions of \mathbf{e}' . Thus \mathbf{e}' is symmetric if and only if \mathbf{e} is.
- (iv) $\text{rk } C(\mathbf{e}, 0) = \text{rk } C(\mathbf{e}', 0)$.

Furthermore, we have $|G_n| = |L_n|$ for every n , and $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$.

Proof. Observe that there exists a permutation $\beta \in S_p$, in fact only one, that normalizes the subgroup $\langle \sigma \rangle$ and such that $\beta(k) = 1$ and $\beta(p) = p$. Indeed, since $\sigma^\beta = (\beta(1) \dots \beta(p))$ and the positions of 1 and p are already fixed in this last tuple, there is only one way to choose the rest of the images of β if we want to obtain a power of σ . Let r be defined by the condition $\sigma^\beta = \sigma^r$, and set $\alpha = \beta^{-1}$. Note that $\alpha(1) = k$ and that, by Lemma 2.15, if $\alpha(i) = j$ then $\alpha(p-i) = p-j$.

Now we define an automorphism f of \mathcal{T} by choosing the labels at all vertices of \mathcal{T} equal to β . We claim that $L = G^f$ satisfies the properties of the statement of the theorem. We have

$$(g^f)_{(v)} = \beta^{-1} g_{(f^{-1}(v))} \beta$$

for every $g \in G$ and every vertex v of the tree. It readily follows that $a^f = a^r$. We now consider $c = b^f$. Let S be the set of all vertices of the form $p.^n.pi$, where $n \geq 0$ and $1 \leq i \leq p-1$. If $v \in S$, then we have $f(v) = p.^n.p\beta(i)$, and consequently $f^{-1}(v) = p.^n.p\alpha(i)$. Thus

$$c_{(v)} = \beta^{-1} b_{(p.^n.p\alpha(i))} \beta = (\sigma^{e_{\alpha(i)}})^\beta = \sigma^{re_{\alpha(i)}}$$

in this case. On the other hand, if $v \notin S$, then also $f^{-1}(v) \notin S$, and so we have $b_{(f^{-1}(v))} = 1$ and $c_{(v)} = 1$. Thus c is the automorphism given by the recursive relation

$$\psi(c) = (a^{re_{\alpha(1)}}, \dots, a^{re_{\alpha(p-1)}}, c).$$

Now, let ℓ be the inverse of $re_{\alpha(1)}$ modulo p , and put $b' = c^\ell$. Then $L = \langle a, b' \rangle$, where b' is the automorphism defined by

$$\psi(b') = (a^{e'_1}, \dots, a^{e'_{p-1}}, b'),$$

i.e. L is the GGS-group with defining vector \mathbf{e}' . This proves (i), (ii), and (iii).

Let us now check (iv). If $C = C(\mathbf{e}, 0)$, $C' = C(\mathbf{e}', 0)$ and we define $e_p = 0$, then

$$c'_{ij} = e_{\alpha(j-i+1)}/e_k = e_{\alpha(j)-\alpha(i)+\alpha(1)}/e_k = c_{\alpha(i)-\alpha(1)+1, \alpha(j)}/e_k,$$

since we know that α is a homomorphism by the proof of Lemma 2.15. (Here, all indices are taken modulo p between 1 and p .) By observing that the maps $i \mapsto \alpha(i) - \alpha(1) + 1$ and $j \mapsto \alpha(j)$ are permutations of \mathbb{F}_p , we conclude that $\text{rk } C = \text{rk } C'$.

Finally, note that, since G and L are conjugate, we clearly have $|G_n| = |L_n|$, and then by (2), also $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$. \square

We want to stress the fact that the automorphism f conjugating G to L in the previous theorem has constant portrait. This has nice consequences, such as the following one.

Proposition 2.17. *Let J and K be two subgroups of $\text{Aut } \mathcal{T}$, where J is contained in $\text{Stab}(1)$. If $f \in \text{Aut } \mathcal{T}$ has constant portrait, then we have*

$$K \times \dots \times K \subseteq \psi(J)$$

if and only if

$$K^f \times \dots \times K^f \subseteq \psi(J^f).$$

Proof. Since f^{-1} is also an automorphism of constant portrait, it suffices to prove the ‘only if’ part. Let β be the permutation appearing at all labels of f . Then we can write $f = ch$, where c is the rooted automorphism corresponding to β and $h \in \text{Stab}(1)$ is such that $\psi(h) = (f, \dots, f)$.

Let us now consider an arbitrary tuple (k_1, \dots, k_p) , with $k_i \in K$ for every $i = 1, \dots, p$. By hypothesis, there exists $j \in J$ such that $\psi(j) = (k_1, \dots, k_p)$. Then $\psi(j^c) = (k_{\beta^{-1}(1)}, \dots, k_{\beta^{-1}(p)})$, and consequently

$$\psi(j^f) = \psi(j^c)^{\psi(h)} = (k_{\beta^{-1}(1)}, \dots, k_{\beta^{-1}(p)})^{(f, \dots, f)} = (k_{\beta^{-1}(1)}^f, \dots, k_{\beta^{-1}(p)}^f).$$

Clearly, this implies that $K^f \times \dots \times K^f \subseteq \psi(J^f)$. \square

The previous proposition will be useful when we want to find a branch structure in a GGS-group. The same can be said about the following result.

Proposition 2.18. *Let G be a GGS-group, and let L and N be two normal subgroups of G . If $L = \langle X \rangle^G$ for a subset X of G , and $(x, 1, \dots, 1) \in \psi(N)$ for every $x \in X$, then*

$$L \times \dots \times L \subseteq \psi(N).$$

Proof. By Proposition 2.2, if $g \in G$ there exists $h \in \text{Stab}_G(1)$ such that the first component of $\psi(h)$ is g . Since $(x, 1, \dots, 1) \in \psi(N)$ and N is normal in G , it follows that $(x^g, 1, \dots, 1) \in \psi(N)$ for every $x \in X$ and $g \in G$. Hence

$$L \times \{1\} \times \dots \times \{1\} \subseteq \psi(N),$$

since $L = \langle x^g \mid x \in X, g \in G \rangle$.

Now, if $\psi(n) = (\ell_1, \ell_2, \dots, \ell_p)$ then $\psi(n^a) = (\ell_p, \ell_1, \dots, \ell_{p-1})$. As a consequence,

$$\{1\} \times \dots \times \{1\} \times L \times \{1\} \times \dots \times \{1\} \subseteq \psi(N),$$

where L may appear at any position. The result follows. \square

3. GGS-GROUPS WITH NON-CONSTANT DEFINING VECTOR

In this section we prove Theorems A and B in the case that the defining vector \mathbf{e} of the GGS-group G is not constant. As it turns out, the key is to prove that G has a certain branch structure. We begin by recalling the concepts that we will need about branching in $\text{Aut } \mathcal{T}$.

Definition 3.1. Let G be a self-similar spherically transitive group of automorphisms of a regular tree, and let K be a non-trivial subgroup of $\text{Stab}_G(1)$. We say that G is *weakly regular branch* over K if

$$K \times \dots \times K \subseteq \psi(K).$$

If furthermore K has finite index in G , we say that G is *regular branch* over K .

It is well-known (and an immediate consequence of Proposition 2.2) that every GGS-group G is self-similar and spherically transitive. We next see that, if \mathbf{e} is not constant, then G is regular branch over $\gamma_3(G)$.

Lemma 3.2. *Let G be a GGS-group with non-constant defining vector. Then*

$$\psi(\gamma_3(\text{Stab}_G(1))) = \gamma_3(G) \times \overset{p}{\dots} \times \gamma_3(G).$$

In particular,

$$\gamma_3(G) \times \overset{p}{\dots} \times \gamma_3(G) \subseteq \psi(\gamma_3(G)),$$

and G is a regular branch group over $\gamma_3(G)$.

Proof. Since $\psi(\text{Stab}_G(1))$ is contained in $G \times \overset{p}{\dots} \times G$, it clearly suffices to prove the inclusion \supseteq . By Theorem 2.16 and Proposition 2.17, we may assume that $\mathbf{e} = (1, e_2, \dots, e_{p-1})$. If $e_{p-1} = 0$ then

$$\psi(b) = (a, \dots, a^{e_{p-2}}, 1, b),$$

and consequently

$$\psi([b_0, b_1, b_0]) = ([a, b, a], 1, \dots, 1)$$

and

$$\psi([b_0, b_1, b_1]) = ([a, b, b], 1, \dots, 1).$$

Since $G = \langle a, b \rangle$, it follows that $\gamma_3(G) = \langle [a, b, a], [a, b, b] \rangle^G$, and then by Proposition 2.18, we have $\gamma_3(G) \times \dots \times \gamma_3(G) \subseteq \psi(\gamma_3(\text{Stab}_G(1)))$. Thus we may assume that $e_{p-1} \neq 0$.

Now we consider the following two cases separately:

- (i) There exists $k \in \{2, \dots, p-2\}$ such that (e_{k-1}, e_k) and (e_k, e_{k+1}) are not proportional.
- (ii) (e_{k-1}, e_k) and (e_k, e_{k+1}) are proportional for all $k = 2, \dots, p-2$.

Observe that if $p = 3$ then case (ii) vacuously holds.

(i) Let us put

$$g_k = b_{p-k+1}^{e_k} b_{p-k}^{-e_{k-1}}$$

for $2 \leq k \leq p-2$, so that

$$\psi(g_k) = (a^{e_k^2 - e_{k-1}e_{k+1}}, \dots, 1).$$

(The intermediate values represented by the dots are not necessarily 1 in this case.) Since (e_{k-1}, e_k) and (e_k, e_{k+1}) are not proportional, we have $e_k^2 - e_{k-1}e_{k+1} \neq 0$. Hence there is a power g of g_k such that

$$\psi(g) = (a, \dots, 1).$$

On the other hand, since

$$\psi(b_1 b_{p-1}^{-e_{p-1}}) = (ba^{-e_2 e_{p-1}}, \dots, 1),$$

with the help of g we can get an element $h \in \text{Stab}_G(1)$ such that

$$\psi(h) = (b, \dots, 1).$$

Consequently,

$$\psi([b_0, b_1, g]) = ([a, b, a], 1, \dots, 1)$$

and

$$\psi([b_0, b_1, h]) = ([a, b, b], 1, \dots, 1),$$

and the result follows as before from Proposition 2.18.

(ii) Since $e_1 = 1$, it follows that $e_i = e_2^{i-1}$ for every $i = 1, \dots, p-1$. (Note that this is valid all the same if $p = 3$.) Hence $\mathbf{e} = (1, m, m^2, \dots, m^{p-2})$ with $m \neq 1$, because \mathbf{e} is not constant. Since $e_{p-1} \neq 0$, we also have $m \neq 0$, and consequently $m^{p-1} = 1$. Then

$$\psi(b_0 b_1^{-m}) = (ab^{-m}, 1, \dots, 1, ba^{-1})$$

and

$$\psi(b_1 b_2^{-m}) = (ba^{-1}, ab^{-m}, 1, \dots, 1).$$

Hence

$$\psi([b_0, b_1, b_1 b_2^{-m}]) = ([a, b, ba^{-1}], 1, \dots, 1)$$

and

$$\psi([b_2^m, b_1, b_0 b_1^{-m}]) = ([a, b, ab^{-m}], 1, \dots, 1).$$

Now, since $G' = \langle [a, b] \rangle^G$ and $\langle ab^{-m}, ba^{-1} \rangle = \langle b^{1-m}, ba^{-1} \rangle$ is the whole of G (at this point, it is essential that $m \neq 1$), it follows that

$$\gamma_3(G) = \langle [a, b, ab^{-m}], [a, b, ba^{-1}] \rangle^G.$$

Thus the result is again a consequence of Proposition 2.18. \square

As a consequence of the previous lemma, we can show that, for \mathbf{e} non-constant and $n \geq 3$, there is a close relation between $\text{Stab}_G(n)$ and $\text{Stab}_G(n-1)$ in a GGS-group G .

Lemma 3.3. *Let G be a GGS-group with non-constant defining vector \mathbf{e} . Then, for every $n \geq 3$ we have*

$$\psi(\text{Stab}_G(n)) = \text{Stab}_G(n-1) \times \cdot^p \times \text{Stab}_G(n-1)$$

and

$$\psi_{n+1}(\text{Stab}_{G_{n+1}}(n)) = \text{Stab}_{G_n}(n-1) \times \cdot^p \times \text{Stab}_{G_n}(n-1).$$

Proof. Clearly, it suffices to prove the first equality. By using Corollary 2.5 and Lemma 3.2, we have

$$\text{Stab}_G(2) \times \cdot^p \times \text{Stab}_G(2) \subseteq \gamma_3(G) \times \cdot^p \times \gamma_3(G) = \psi(\gamma_3(\text{Stab}_G(1))).$$

Thus $\text{Stab}_G(n-1) \times \cdots \times \text{Stab}_G(n-1)$ is contained in the image of $\text{Stab}_G(1)$ under ψ for all $n \geq 3$, and the result follows. \square

If the vector \mathbf{e} is non-symmetric, we can improve Lemma 3.2 as follows.

Lemma 3.4. *Let G be a GGS-group with non-symmetric defining vector. Then*

$$\psi(\text{Stab}_G(1)') = G' \times \cdot^p \times G'.$$

In particular,

$$G' \times \cdot^p \times G' \subseteq \psi(G'),$$

and G is a regular branch group over G' .

Proof. Observe that we only need to care about the inclusion \supseteq . By Theorem 2.16 and Proposition 2.17, we may assume that $e_1 = 1$ and $e_{p-1} \neq 1$, since \mathbf{e} is non-symmetric. Let us write m for e_{p-1} .

By using (3), we get

$$\begin{aligned} \psi([b_0, b_1]) &= ([a, b], 1, \dots, 1, [b, a^m]) \\ &\equiv ([a, b], 1, \dots, 1, [a, b]^{-m}) \pmod{\gamma_3(G) \times \cdot^p \times \gamma_3(G)}, \\ \psi([b_{p-1}, b_0]^m) &= (1, \dots, 1, [b, a^m]^m, [a, b]^m) \\ &\equiv (1, \dots, 1, [a, b]^{-m^2}, [a, b]^m) \pmod{\gamma_3(G) \times \cdot^p \times \gamma_3(G)}, \\ &\vdots \\ \psi([b_1, b_2]^{m^{p-1}}) &= ([b, a^m]^{m^{p-1}}, [a, b]^{m^{p-1}}, 1, \dots, 1) \\ &\equiv ([a, b]^{-m^p}, [a, b]^{m^{p-1}}, 1, \dots, 1) \pmod{\gamma_3(G) \times \cdot^p \times \gamma_3(G)}. \end{aligned}$$

Since $m^p = m$ (recall that $m \in \mathbb{F}_p$), if we multiply together all the expressions above, we obtain that

$$\begin{aligned} \psi([b_0, b_1][b_{p-1}, b_0]^m \dots [b_1, b_2]^{m^{p-1}}) &\equiv ([a, b]^{1-m}, 1, \dots, 1) \\ &\pmod{\gamma_3(G) \times \cdot^p \times \gamma_3(G)}. \end{aligned}$$

If we use the inclusion

$$\gamma_3(G) \times \cdot^p \times \gamma_3(G) \subseteq \psi(\text{Stab}_G(1)'),$$

which is a consequence of Lemma 3.2, we get

$$([a, b]^{1-m}, 1, \dots, 1) \in \psi(\text{Stab}_G(1)').$$

Now, since $G = \langle a, b \rangle$ and $m \neq 1$, it follows that G' is the normal closure of $[a, b]^{1-m}$. By Proposition 2.18, we conclude that $G' \times \cdots \times G' \subseteq \psi(\text{Stab}_G(1)')$. \square

Now we can proceed to calculate the order of G_n for every $n \geq 1$, and as a consequence, to obtain the Hausdorff dimension of \overline{G} in Γ , provided that the defining vector \mathbf{e} is not constant. We deal separately with the following two cases: (i) \mathbf{e} is not symmetric; (ii) \mathbf{e} is symmetric and not constant. In both cases, the key is to determine the order of $\text{Stab}_{G_3}(2)$ and to use Lemma 3.3. We begin by case (i).

Theorem 3.5. *Let G be a GGS-group with non-symmetric defining vector \mathbf{e} . Then*

$$|\text{Stab}_{G_3}(2)| = p^{t(p-1)},$$

where t is the rank of $C(\mathbf{e}, 0)$.

Proof. By Theorem 2.16, we may assume that $e_1 = 1$ and $e_{p-1} \neq 1$. For simplicity, let us write C for $C(\mathbf{e}, 0)$. Since $\text{Stab}_{G_3}(2) = \text{Stab}_G(2)/\text{Stab}_G(3)$, we are going to study the image of $\text{Stab}_G(2)$ under the canonical epimorphism π from G onto G_3 .

Let g be an arbitrary element of $\text{Stab}_G(1)$, and let (r_0, \dots, r_{p-1}) denote the weight vector of g . By Theorem 2.13, we have $g \in \text{Stab}_G(2)$ if and only if

$$(r_0 \ r_1 \ \dots \ r_{p-1})C = (0 \ 0 \ \dots \ 0).$$

Since the rank of C is t , this system has p^{p-t} solutions, which we denote by

$$r^{(i)} = (r_0^{(i)}, \dots, r_{p-1}^{(i)}),$$

for $i = 1, \dots, p^{p-t}$. We may assume that $r^{(1)} = (0, \dots, 0)$.

Each solution $r^{(i)}$ determines a subset $R^{(i)}$ of $\text{Stab}_G(2)$, consisting of all the elements whose weight vector is $r^{(i)}$. Put $S^{(i)} = \pi(R^{(i)})$. By the discussion in the previous paragraph, we know that $\text{Stab}_{G_3}(2)$ is the union of all the $S^{(i)}$ for $i = 1, \dots, p^{p-t}$. We will prove the following:

- (i) If $i \neq j$ then $S^{(i)}$ and $S^{(j)}$ are disjoint. (By Theorem 2.8, we know that $R^{(i)}$ and $R^{(j)}$ are disjoint, but we have to rule out the possibility that an element in $R^{(i)}$ and an element in $R^{(j)}$ have the same image in G_3 .)
- (ii) $|S^{(i)}| = p^{p(t-1)}$ for all $i = 1, \dots, p^{p-t}$.

Once (i) and (ii) are proved, it readily follows that $|\text{Stab}_{G_3}(2)| = p^{t(p-1)}$, as desired.

We begin by proving (i). For this purpose, assume that $g \in R^{(i)}$ and $h \in R^{(j)}$ are two elements with the same image in G_3 . Then $gh^{-1} \in \text{Stab}_G(3)$ and, by Theorem 2.13, the weight vector of gh^{-1} is $(0, \dots, 0)$. Since the weight vector defines a homomorphism from $\text{Stab}_G(1)$ to \mathbb{F}_p^p , it follows that $r^{(i)} = r^{(j)}$, and so $i = j$, as desired.

Now we proceed to the proof of (ii). By definition, each $S^{(i)}$ is non-empty. If h_i is an element of $S^{(i)}$, then it is clear that $S^{(i)} = h_i S^{(1)}$. Thus $|S^{(i)}| = |S^{(1)}|$, and it suffices to see that $S^{(1)}$ has the desired cardinality. Let g be an arbitrary element of $\text{Stab}_G(2)$. According to (6), we have $g \in R^{(1)}$ if and only if each component of $\psi(g)$ has total weight equal to 0. By Theorem

2.11, this is equivalent to $\psi(g)$ lying in $G' \times \cdots \times G'$. On the other hand, since $G' \leq \text{Stab}_G(1)$, we have $\psi^{-1}(G' \times \cdots \times G') \leq \text{Stab}(2)$. Hence

$$(9) \quad R^{(1)} = G \cap \psi^{-1}(G' \times \cdots \times G').$$

Note that this equality is valid for any defining vector \mathbf{e} . Now, since we are working under the assumption that \mathbf{e} is non-symmetric, we have $G' \times \cdots \times G' \leq \psi(G')$ by Lemma 3.4. Thus we conclude that $R^{(1)} = \psi^{-1}(G' \times \cdots \times G')$ in this case or, equivalently, that

$$\psi(R^{(1)}) = G' \times \cdots \times G'.$$

We consider now the following commutative diagram:

$$(10) \quad \begin{array}{ccc} R^{(1)} & \xrightarrow{\pi} & S^{(1)} \\ \psi \downarrow & & \downarrow \psi_3 \\ G' \times \cdots \times G' & \xrightarrow{\tilde{\pi} \times \cdots \times \tilde{\pi}} & \frac{G'}{\text{Stab}_G(2)} \times \cdots \times \frac{G'}{\text{Stab}_G(2)}, \end{array}$$

where $\tilde{\pi}$ denotes reduction modulo $\text{Stab}_G(2)$. (Take into account that G' contains $\text{Stab}_G(2)$ by Theorem 2.1.) By the discussion of the preceding paragraph, the left vertical arrow of the diagram is surjective. Consequently, the right vertical arrow is also surjective, and since it is obviously injective, it follows that it is a bijective map. In particular,

$$|S^{(1)}| = |G' : \text{Stab}_G(2)|^p.$$

Now, by Theorems 2.1 and 2.4, we have $|G : G'| = p^2$ and $|G : \text{Stab}_G(2)| = p^{t+1}$. Thus $|G' : \text{Stab}_G(2)| = p^{t-1}$, and we conclude that $|S^{(1)}| = p^{p(t-1)}$, as desired. \square

Theorem 3.6. *Let G be a GGS-group with non-symmetric defining vector \mathbf{e} . Then*

$$\log_p |G_n| = tp^{n-2} + 1, \quad \text{for every } n \geq 2,$$

where t is the rank of $C(\mathbf{e}, 0)$, and

$$\dim_{\Gamma} \overline{G} = \frac{(p-1)t}{p^2}.$$

Proof. We argue by induction on $n \geq 2$. By Theorem 2.4, we have $|G_2| = p^{t+1}$. Suppose now that $n > 2$ and that the result is true for $n-1$. By using Lemma 3.3, we have

$$|\text{Stab}_{G_n}(n-1)| = |\text{Stab}_{G_{n-1}}(n-2)|^p = \cdots = |\text{Stab}_{G_3}(2)|^{p^{n-3}}.$$

Since $|\text{Stab}_{G_3}(2)| = p^{t(p-1)}$ by Theorem 3.5, we conclude that

$$|G_n| = |G_{n-1}| |\text{Stab}_{G_n}(n-1)| = p^{tp^{n-3}+1} \cdot p^{tp^{n-3}(p-1)} = p^{tp^{n-2}+1},$$

as desired. Finally, the value of $\dim_{\Gamma} \overline{G}$ follows directly from (2). \square

Next we consider the case when the vector \mathbf{e} is non-constant and symmetric.

Theorem 3.7. *Let G be a GGS-group with symmetric non-constant defining vector \mathbf{e} . Then*

$$|\text{Stab}_{G_3}(2)| = p^{t(p-1)-1},$$

where t is the rank of $C(\mathbf{e}, 0)$.

Proof. Let π , $R^{(i)}$ and $S^{(i)}$ for $i = 1, \dots, p^{p-t}$ be as in the proof of Theorem 3.5. The plan of the proof is the same as in that theorem. The difference is that, in this case, we need to see that

$$|S^{(1)}| = p^{p(t-1)-1}.$$

For that purpose, it suffices to prove that the image of $S^{(1)}$ under the injective map ψ_3 is a subgroup of index p of

$$\frac{G'}{\text{Stab}_G(2)} \times \cdots \times \frac{G'}{\text{Stab}_G(2)}.$$

We know from (9) that $R^{(1)} = G \cap \psi^{-1}(G' \times \cdots \times G')$ consists of all elements of G whose weight vector is $(0, \dots, 0)$. According to Theorem 2.14, we have $R^{(1)} = \text{Stab}_G(1)'$. Hence

$$(11) \quad R^{(1)} = \langle [b_i, b_j]^h \mid 0 \leq i, j \leq p-1, h \in \text{Stab}_G(1) \rangle.$$

Let us consider again the commutative diagram in (10). Since

$$\ker(\tilde{\pi} \times \cdots \times \tilde{\pi}) = \text{Stab}_G(2) \times \cdots \times \text{Stab}_G(2) = \psi(\text{Stab}_G(3))$$

by Lemma 3.3, and since $\text{Stab}_G(3) \leq R^{(1)}$ by Theorem 2.13, it follows that the index

$$\left| \frac{G'}{\text{Stab}_G(2)} \times \cdots \times \frac{G'}{\text{Stab}_G(2)} : \psi_3(S^{(1)}) \right| = \left| (\tilde{\pi} \times \cdots \times \tilde{\pi})(G' \times \cdots \times G') : (\tilde{\pi} \times \cdots \times \tilde{\pi})(\psi(R^{(1)})) \right|$$

is the same as

$$|G' \times \cdots \times G' : \psi(R^{(1)})|.$$

Thus it suffices to prove that this last index is p .

Let $\bar{\psi}$ the map from $R^{(1)}$ to $G'/\gamma_3(G) \times \cdots \times G'/\gamma_3(G)$ which is obtained by first applying ψ and then reducing every component modulo $\gamma_3(G)$. Observe that $G'/\gamma_3(G) \times \cdots \times G'/\gamma_3(G)$ can be seen as a vector space of dimension p over \mathbb{F}_p , since $|G' : \gamma_3(G)| = p$. Since we may assume that $e_1 = 1$, and since $e_{p-1} = e_1$, we have

$$\psi([b_i, b_{i+1}]) = (1, \dots, 1, [b, a], [a, b], 1, \dots, 1), \quad \text{for } i = 1, \dots, p-1,$$

where $[b, a]$ appears at the i th position. Now, $G'/\gamma_3(G)$ is generated by the image of $[b, a]$, and so it readily follows that the dimension of $\bar{\psi}(R^{(1)})$ is at least $p-1$. Hence

$$|G' \times \cdots \times G' : \psi(R^{(1)})| = 1 \text{ or } p.$$

Since $\gamma_3(G) \times \cdots \times \gamma_3(G) \leq \psi(R^{(1)})$ by Lemma 3.2 and (9), we get

$$|G' \times \cdots \times G' : \psi(R^{(1)})| = 1 \text{ or } p.$$

Thus it suffices to see that $([a, b], 1, \dots, 1) \notin \psi(R^{(1)})$ in order to conclude that $|G' \times \cdots \times G' : \psi(R^{(1)})| = p$, as desired.

Let $\lambda : \text{Stab}_G(1) \longrightarrow \mathbb{F}_p$ be the homomorphism given by

$$g \longmapsto \sum_{i=0}^{p-1} ir_i,$$

where (r_0, \dots, r_{p-1}) is the weight vector of g . If $g \in \text{Stab}_G(1)$ then the weight vector of g^b is also (r_0, \dots, r_{p-1}) , and the weight vector of g^a is $(r_{p-1}, r_0, \dots, r_{p-2})$. Hence $\lambda(g^b) = \lambda(g)$, and if $g \in G'$, then furthermore

$$\lambda(g^a) = \sum_{i=0}^{p-1} ir_{i-1} = \sum_{i=0}^{p-1} r_{i-1} + \sum_{i=0}^{p-1} (i-1)r_{i-1} = \lambda(g),$$

since $r_0 + \dots + r_{p-1} = 0$ by Theorem 2.11. It follows that $\lambda(g^h) = \lambda(g)$ for every $g \in G'$ and $h \in G$.

Now we define $\Lambda : G' \times \dots \times G' \longrightarrow \mathbb{F}_p$ by means of

$$\Lambda(g_1, \dots, g_p) = \lambda(g_1) + \dots + \lambda(g_p).$$

By the preceding paragraph, we have

$$\Lambda(g^h) = \Lambda(g), \quad \text{for all } g \in G' \times \dots \times G' \text{ and } h \in G \times \dots \times G.$$

Hence $\ker \Lambda$ is a normal subgroup of $G \times \dots \times G$.

For every $1 \leq i < j \leq p$, we have

$$\begin{aligned} \psi([b_i, b_j]) &= (1, \dots, 1, [b, a^{e_{i-j}}], 1, \dots, 1, [a^{e_{j-i}}, b], 1, \dots, 1) = \\ &= (1, \dots, 1, b_0^{-1} b_{e_{i-j}}, 1, \dots, 1, b_{e_{j-i}}^{-1} b_0, 1, \dots, 1), \end{aligned}$$

where the non-trivial components are at positions i and j . Since \mathbf{e} is symmetric, we have $e_{i-j} = e_{j-i}$, and consequently

$$\Lambda(\psi([b_i, b_j])) = e_{i-j} - e_{j-i} = 0.$$

Hence $\psi([b_i, b_j]) \in \ker \Lambda$, and since $\ker \Lambda$ is a normal subgroup of $G \times \dots \times G$, it follows from (11) that $\psi(R^{(1)}) \leq \ker \Lambda$. Since

$$\Lambda([a, b], 1, \dots, 1) = \Lambda(b_1^{-1} b_0, 1, \dots, 1) = -1,$$

we deduce that $([a, b], 1, \dots, 1) \notin \psi(R^{(1)})$, which completes the proof. \square

Theorem 3.8. *Let G be a GGS-group with a non-constant symmetric defining vector \mathbf{e} . Then*

$$\log_p |G_n| = tp^{n-2} + 1 - \frac{p^{n-2} - 1}{p - 1}, \quad \text{for every } n \geq 2,$$

where t is the rank of $C(\mathbf{e}, 0)$, and

$$\dim_{\Gamma} \overline{G} = \frac{(p-1)t-1}{p^2}.$$

Proof. The proof is completely similar to that of Theorem 3.6. \square

4. GGS-GROUPS WITH CONSTANT DEFINING VECTOR

In this section, we deal with the case where the defining vector is constant, say $\mathbf{e} = (e, \dots, e)$, where $e \in \mathbb{F}_p^\times$. Let m be the inverse of e in \mathbb{F}_p^\times , and $b^* = b^m$. Then $G = \langle a, b^* \rangle$, and $\psi(b^*) = (a, \dots, a, b^*)$. For this reason, we may assume in the remainder of this section that $\mathbf{e} = (1, \dots, 1)$.

We begin by defining a sequence of elements of G that will be fundamental in the sequel. We put $y_0 = ba^{-1}$ and, more generally, $y_i = y_0^{a^i}$ for every integer i . Thus $y_i^{a^j} = y_{i+j}$ for all $i, j \in \mathbb{Z}$. Also,

$$(12) \quad y_i^b = y_i^{aa^{-1}b} = y_{i+1}^{y_1}.$$

Observe that $y_i = y_j$ if $i \equiv j \pmod{p}$, so that the set $\{y_0, \dots, y_{p-1}\}$ already contains all the y_i . In the following lemma, we collect some important properties of the elements y_i . We adopt the following convention: given a vector v of length p and an integer i , not lying in the range $\{1, \dots, p\}$, the i th position of v is to be understood as the j th position, where $j \in \{1, \dots, p\}$ and $i \equiv j \pmod{p}$.

Lemma 4.1. *Let G be a GGS-group with constant defining vector. Then:*

- (i) $y_{p-1}y_{p-2} \dots y_1y_0 = 1$.
- (ii) *If z_i is the tuple of length p having y_2 at position $i - 2$, y_1^{-1} at position $i - 1$, and 1 elsewhere, then*

$$(13) \quad \psi([y_i, y_j]) = z_i z_j^{-1}, \quad \text{for every } i \text{ and } j.$$

- (iii) *We have*

$$(14) \quad [y_i, y_j] = [y_i, y_{i-1}][y_{i-1}, y_{i-2}] \dots [y_{j+1}, y_j], \quad \text{for every } i > j.$$

Proof. (i) We have

$$\begin{aligned} y_{p-1}y_{p-2} \dots y_1y_0 &= a^{-(p-1)}ba^{p-2} \cdot a^{-(p-2)}ba^{p-3} \dots a^{-1}b \cdot ba^{-1} \\ &= a^{-(p-1)}b^p a^{-1} = 1. \end{aligned}$$

(ii) Clearly, it is enough to see the result for $i > j$. On the other hand, since both sequences $\{y_i\}$ and $\{z_i\}$ are periodic of period p , we may assume that i and j lie in the set $\{3, \dots, p+2\}$. If $r = j - 3$ and $k = i - r$, then

$$[y_i, y_j] = [y_k^{a^r}, y_3^{a^r}] = [y_k, y_3]^{a^r},$$

and so $\psi([y_i, y_j])$ is the result of applying to $\psi([y_k, y_3])$ the permutation which moves every element r positions to the right. It readily follows that it suffices to prove (13) for $[y_k, y_3]$ with $4 \leq k \leq p+2$.

Since $y_i = a^{-i}ba^{i-1} = a^{-1}b_{i-1}$ for every i , we have

$$(15) \quad [y_k, y_3] = b_{k-1}^{-1}ab_2^{-1}b_{k-1}a^{-1}b_2 = b_{k-1}^{-1}b_1^{-1}b_{k-2}b_2 = (b_1^{-1}b_{k-2})^{b_{k-1}}(b_{k-1}^{-1}b_2).$$

Now, it follows from (3) that

$$\begin{aligned} \psi((b_1^{-1}b_{k-2})^{b_{k-1}}) &= (y_1^{-1}, 1, \overset{k-4}{\dots}, 1, y_1, 1, \dots, 1)^{(a, \overset{k-2}{\dots}, a, b, a, \dots, a)} \\ &= \begin{cases} (y_2^{-1}, 1, \overset{k-4}{\dots}, 1, y_2, 1, \dots, 1), & \text{if } 4 \leq k \leq p+1, \\ (y_1^{-1}y_2^{-1}y_1, 1, \dots, 1, y_2), & \text{if } k = p+2. \end{cases} \end{aligned}$$

Here, we have used that $y_1^b = y_2^{y_1}$ by (12). Similarly,

$$\psi(b_{k-1}^{-1}b_2) = \begin{cases} (1, y_1, 1, \dots, 1, y_1^{-1}, 1, \dots, 1), & \text{if } 4 \leq k \leq p+1, \\ (y_1^{-1}, y_1, 1, \dots, 1), & \text{if } k = p+2. \end{cases}$$

By taking these values to (15), we obtain that $\psi([y_k, y_3]) = z_k z_3^{-1}$, as desired.

(iii) This follows immediately from (ii), since

$$\begin{aligned} \psi([y_i, y_j]) &= (z_i z_{i-1}^{-1})(z_{i-1} z_{i-2}^{-1}) \dots (z_{j+1} z_j^{-1}) \\ &= \psi([y_i, y_{i-1}]) \psi([y_{i-1}, y_{i-2}]) \dots \psi([y_{j+1}, y_j]) \\ &= \psi([y_i, y_{i-1}][y_{i-1}, y_{i-2}] \dots [y_{j+1}, y_j]). \end{aligned}$$

□

Next we introduce a maximal subgroup K of G that will play a key role in the determination of the order of G_n in the case that \mathbf{e} is constant.

Lemma 4.2. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. Then:*

- (i) $G' \leq K$ and $|G : K| = p$.
- (ii) $K = \langle y_0, y_1, \dots, y_{p-1} \rangle$ and $K' = \langle [y_1, y_0] \rangle^G$.
- (iii) $K' \times \dots \times K' \subseteq \psi(K') \subseteq \psi(G') \subseteq K \times \dots \times K$. In particular, G is a weakly regular branch group over K' .
- (iv) If $L = \psi^{-1}(K' \times \dots \times K')$ (which, by (iii), is contained in K'), then the conjugates $[y_{i+1}, y_i]^{b^j}$, where $0 \leq i, j \leq p-1$, generate K' modulo L .

Proof. (i) Since $[a, ba^{-1}] = [a, b]^{a^{-1}} \in K$ and K is normal in G , it follows that G' is contained in K . Then $|G : K| = |G/G' : K/G'| = p$.

(ii) Let us first prove that $K = \langle y_0, y_1, \dots, y_{p-1} \rangle$. For this purpose, it suffices to see that $N = \langle y_0, y_1, \dots, y_{p-1} \rangle$ is a normal subgroup of G . This is clear, since $y_i^a = y_{i+1}$ and $y_i^b = y_{i+1}^{y_1}$ for every i .

It follows that

$$K' = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1 \rangle^K = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1 \rangle^G,$$

where the second equality holds because K' is normal in G . By (14), every commutator $[y_i, y_j]$ with $0 \leq j < i \leq p-1$ can be expressed in terms of the $[y_k, y_{k-1}]$ with $k = 1, \dots, p-1$. Since $[y_k, y_{k-1}] = [y_1, y_0]^{a^{k-1}}$, we conclude that $K' = \langle [y_1, y_0] \rangle^G$.

(iii) Let us first prove the inclusion $\psi(G') \subseteq K \times \dots \times K$. We have

$$\begin{aligned} \psi([b, a]) &= \psi(b^{-1}b^a) = (a^{-1}, a^{-1}, \dots, a^{-1}, b^{-1})(b, a, \dots, a, a) \\ &= (a^{-1}b, 1, \dots, 1, b^{-1}a) \in K \times \dots \times K. \end{aligned}$$

Now, since K is normal in G , it readily follows that

$$\psi([b, a]^g) \in K \times \dots \times K, \quad \text{for every } g \in G.$$

This proves the desired inclusion.

Now we focus on proving that $K' \times \dots \times K' \subseteq \psi(K')$. By Proposition 2.18 and (ii), it suffices to see that

$$([y_1, y_0], 1, \dots, 1) \in \psi(K').$$

We consider separately the cases $p \geq 5$ and $p = 3$.

Suppose first that $p \geq 5$. By using (13), we have

$$\psi([y_1, y_2]) = (y_1, 1, \dots, 1, y_2, y_1^{-1}y_2^{-1})$$

and

$$\psi([y_3, y_4]) = (y_2, y_1^{-1}y_2^{-1}, y_1, 1, \dots, 1).$$

If $k = [[y_3, y_4], [y_1, y_2]]$, it follows that

$$\psi(k) = ([y_2, y_1], 1, \dots, 1),$$

since $p \geq 5$. Hence

$$([y_1, y_0], 1, \dots, 1) = \psi(k^{b^{-1}}) \in \psi(K'),$$

as desired.

Assume now that $p = 3$. We have

$$\psi([y_1, y_0]) = (y_1y_0, y_0^{-1}, y_1^{-1}),$$

since $y_2y_1y_0 = 1$, by (i) of Lemma 4.1. Hence

$$\begin{aligned} \psi([y_0, y_1]^b) &= (y_0^{-1}y_1^{-1}, y_0, y_1)^{(a,a,b)} = (y_1^{-1}y_2^{-1}, y_1, y_1^b) \\ &= ((y_2y_1)^{-1}, y_1, y_2^{y_1}) = (y_0, y_1, (y_0^{-1}y_1^{-1})^{y_1}) \\ &= (y_0, y_1, y_1^{-1}y_0^{-1}), \end{aligned}$$

and

$$([y_1, y_0], 1, 1) = \psi([y_0, y_1]^{ba}[y_1, y_0]) \in \psi(K'),$$

which completes the proof.

(iv) Let us consider an arbitrary element $g \in G$, and let us write $g = ha^ib^j$, for some $i, j \in \mathbb{Z}$, $h \in G'$. Then

$$[y_1, y_0]^g = ([y_1, y_0][y_1, y_0, h])^{a^ib^j} \equiv [y_1, y_0]^{a^ib^j} = [y_{i+1}, y_i]^{b^j} \pmod{L},$$

since $\psi([y_1, y_0, h]) \in \psi(G'') \subseteq K' \times \dots \times K'$ by (iii). Now, since the conjugates $[y_1, y_0]^g$ generate K' by (ii), the result follows. \square

In the following results, we consider the action of an element of G by conjugation as an endomorphism of K/K' , which allows us to multiply several conjugates of an element of K , modulo K' , by adding the elements by which we are conjugating. This gives a meaning to expressions like $g^{1+a+\dots+a^{p-1}} \in K'$ for an element $g \in K$.

Lemma 4.3. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. If $g \in K$ then*

$$g^{1+a+\dots+a^{p-1}} \in K'.$$

Proof. The map R sending $g \in K$ to $g^{1+a+\dots+a^{p-1}}K'$ is a well-defined homomorphism from K to K/K' , and we want to see that R is the trivial homomorphism. Since $K = \langle y_0, \dots, y_{p-1} \rangle$ by (ii) of Lemma 4.2, it suffices to check that $y_i \in \ker R$ for every i . Now,

$$R(y_i) = y_i y_{i+1} \dots y_{p-1} y_0 \dots y_{i-1} K' = y_{p-1} y_{p-2} \dots y_1 y_0 K' = K'$$

by (i) of Lemma 4.1, and we are done. \square

Lemma 4.4. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. If $g \in K'$ and we write $\psi(g) = (g_1, \dots, g_p)$, then:*

- (i) $g_p g_{p-1} \dots g_1 \in K'$.
- (ii) $\prod_{i=1}^{p-1} g_i^{a+a^2+\dots+a^i} \in K'$.

Similarly, if $g \in K' \text{Stab}_G(n)$ for some $n \geq 1$, then both $g_p g_{p-1} \dots g_1$ and $\prod_{i=1}^{p-1} g_i^{a+a^2+\dots+a^i}$ lie in $K' \text{Stab}_G(n-1)$.

Proof. We first deal with the case that $g \in K'$. Let us consider the following two maps:

$$\begin{aligned} P &: K \times \overset{p}{\cdot} \times K \longrightarrow K/K' \\ (g_1, \dots, g_p) &\longmapsto g_p \dots g_1 K', \end{aligned}$$

and

$$\begin{aligned} Q &: K \times \overset{p}{\cdot} \times K \longrightarrow K/K' \\ (g_1, \dots, g_p) &\longmapsto \prod_{i=1}^{p-1} g_i^{a+a^2+\dots+a^i} K'. \end{aligned}$$

Clearly, P and Q are homomorphisms. By (iii) of Lemma 4.2, $\psi(K')$ is contained in the domain of P and Q , and our goal is to prove that it is actually in the kernels of these maps. Since the image of $K' \times \overset{p}{\cdot} \times K'$ is trivial, it suffices to see that $\psi(g) \in \ker P$ and $\psi(g) \in \ker Q$ for every g in a system of generators of K' modulo L , where $L = \psi^{-1}(K' \times \overset{p}{\cdot} \times K')$. By (iv) of Lemma 4.2, the conjugates $[y_{i+1}, y_i]^{b^j}$, for $i, j = 0, \dots, p-1$ constitute such a set of generators.

Let $c \in \Gamma$ be defined by means of $\psi(c) = (a, a, \dots, a)$. We claim that

$$(16) \quad g^b \equiv g^c \pmod{L}, \quad \text{for every } g \in K'.$$

Indeed, we have $\psi(b) = \psi(c)(1, \dots, 1, a^{-1}b)$, and so

$$\begin{aligned} \psi(g^b) &= \psi(g^c)^{(1, \dots, 1, a^{-1}b)} = \psi(g^c)[\psi(g^c), (1, \dots, 1, a^{-1}b)] \\ &\equiv \psi(g^c) \pmod{K' \times \overset{p}{\cdot} \times K'}, \end{aligned}$$

since $\psi(g^c) \in K \times \overset{p}{\cdot} \times K$ and $a^{-1}b \in K$.

As a consequence of (16), it suffices to see that $\psi([y_{i+1}, y_i]^{c^j})$ lies in both $\ker P$ and $\ker Q$. Since

$$P(\psi([y_{i+1}, y_i]^{c^j})) = P(\psi([y_{i+1}, y_i]))^{a^j}$$

and

$$Q(\psi([y_{i+1}, y_i]^{c^j})) = Q(\psi([y_{i+1}, y_i]))^{a^j},$$

we have reduced ourselves to proving that $\psi([y_{i+1}, y_i])$ is in the kernel of P and Q for every i . According to (13), we have $\psi([y_{i+1}, y_i]) = z_{i+1} z_i^{-1}$, with z_i as defined in Lemma 4.1. Now, one can easily check that

$$P(z_i) = y_1^{-1} y_2 K' \quad \text{and} \quad Q(z_i) = y_2^{-1} K' \quad \text{for every } i,$$

where in the case of Q and $i = 1$ we need to use that

$$y_2^{a+a^2+\dots+a^{p-1}} \equiv y_2^{-1} \pmod{K'},$$

by Lemma 4.3. It readily follows that $\psi([y_{i+1}, y_i])$ lies in both $\ker P$ and $\ker Q$, as desired.

Assume now that $g \in K' \text{Stab}_G(n)$, and let us write $g = fh$, with $f \in K'$ and $h \in \text{Stab}_G(n)$. Put $\psi(f) = (f_1, \dots, f_p)$ and $\psi(h) = (h_1, \dots, h_p)$. Since $h_1, \dots, h_p \in \text{Stab}_G(n-1)$, which is a normal subgroup of G , we have

$$g_p \dots g_1 = f_p h_p \dots f_1 h_1 = f_p \dots f_1 h^*,$$

for some $h^* \in \text{Stab}_G(n-1)$. Since $f \in K'$, we already know that $f_p \dots f_1 \in K'$, and so we conclude that $g_p \dots g_1 \in K' \text{Stab}_G(n-1)$, as desired. The second assertion can be proved in a similar way. \square

Theorem 4.5. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$ and $L = \psi^{-1}(K' \times \dots \times K')$. Then the following isomorphisms hold:*

$$K'/L \cong K/K' \times \dots \times K/K',$$

and

$$K' \text{Stab}_G(n)/L \text{Stab}_G(n) \cong K/K' \text{Stab}_G(n-1) \times \dots \times K/K' \text{Stab}_G(n-1),$$

for every $n \geq 3$.

Proof. Let π be the map given by

$$\begin{aligned} K \times \dots \times K &\longrightarrow K/K' \times \dots \times K/K' \\ (g_1, \dots, g_p) &\longmapsto (g_1 K', \dots, g_p K'), \end{aligned}$$

and let R be the composition of $\psi : K' \longrightarrow K \times \dots \times K$ with π . If we see that R is surjective, and that $\ker R = L$, then the first isomorphism of the statement follows.

Let $g \in K'$ be an element lying in $\ker R$. If $\psi(g) = (g_1, \dots, g_p)$, then we have $g_1, \dots, g_{p-2} \in K'$. By (ii) of Lemma 4.4, it follows that

$$g_{p-1}^{a+\dots+a^{p-1}} \in K',$$

and by applying Lemma 4.3, we get $g_{p-1} \in K'$. Now, (i) of Lemma 4.4 immediately yields that also $g_p \in K'$. This proves that $\ker R = L$.

Now we prove that

$$(17) \quad K/K' \times \{\bar{1}\} \times \dots \times \{\bar{1}\} \subseteq R(K').$$

Then, by arguing as in the proof of Proposition 2.18, it follows that R is surjective. By (13), we have

$$\psi([y_1, y_2]) = (y_1, 1, \dots, 1, h_{p-1}, h_p)$$

for some elements $h_{p-1}, h_p \in K$. Hence

$$\psi([y_1, y_2]^{b^{i-1}}) = (y_i, 1, \dots, 1, h_{p-1}^*, h_p^*)$$

for every i , and we are done, since $K = \langle y_0, \dots, y_{p-1} \rangle$.

The second isomorphism can be proved in a similar way. Observe that the condition $n \geq 3$ guarantees that $\text{Stab}_G(n-1) \leq G' \leq K$, so that it makes sense to write $K/K' \text{Stab}_G(n-1)$. Consider this time the homomorphism

$$\begin{aligned} \pi_n : K \times \dots \times K &\longrightarrow K/K' \text{Stab}_G(n-1) \times \dots \times K/K' \text{Stab}_G(n-1) \\ (g_1, \dots, g_p) &\longmapsto (g_1 K' \text{Stab}_G(n-1), \dots, g_p K' \text{Stab}_G(n-1)), \end{aligned}$$

and let R_n be the composition of $\psi : K' \longrightarrow K \times \cdot^p \times K$ with π_n . Observe that the surjectiveness of R already implies that R_n is surjective. Let us prove that $\ker R_n = L \operatorname{Stab}_G(n) \cap K'$. The same proof as above, but using the last part of Lemma 4.4, shows that

$$\begin{aligned} \psi(\ker R_n) &= (K' \operatorname{Stab}_G(n-1) \times \cdot^p \times K' \operatorname{Stab}_G(n-1)) \cap \psi(K') \\ &= (K' \times \cdot^p \times K')(\operatorname{Stab}_G(n-1) \times \cdot^p \times \operatorname{Stab}_G(n-1)) \cap \psi(K'). \end{aligned}$$

Since $K' \times \cdot^p \times K' \subseteq \psi(K')$, we can apply Dedekind's Law to get

$$\psi(\ker R_n) = (K' \times \cdot^p \times K')((\operatorname{Stab}_G(n-1) \times \cdot^p \times \operatorname{Stab}_G(n-1)) \cap \psi(K')).$$

Now, since $n \geq 3$, we have

$$\begin{aligned} (\operatorname{Stab}_G(n-1) \times \cdot^p \times \operatorname{Stab}_G(n-1)) \cap \psi(K') &= \psi(\operatorname{Stab}_G(n)) \cap \psi(K') \\ &= \psi(\operatorname{Stab}_G(n) \cap K'), \end{aligned}$$

and it follows that

$$\begin{aligned} \psi(\ker R_n) &= (K' \times \cdot^p \times K')\psi(\operatorname{Stab}_G(n) \cap K') = \psi(L)\psi(\operatorname{Stab}_G(n) \cap K') \\ &= \psi(L(\operatorname{Stab}_G(n) \cap K')). \end{aligned}$$

Hence

$$\ker R_n = L(\operatorname{Stab}_G(n) \cap K') = L \operatorname{Stab}_G(n) \cap K',$$

as claimed.

Now, we can readily obtain the desired isomorphism:

$$\begin{aligned} K' \operatorname{Stab}_G(n) / L \operatorname{Stab}_G(n) &\cong K' / (L \operatorname{Stab}_G(n) \cap K') = K' / \ker R_n \\ &\cong R_n(K') = K / K' \operatorname{Stab}_G(n-1) \times \cdot^{p-2} \times K / K' \operatorname{Stab}_G(n-1). \end{aligned}$$

□

Theorem 4.6. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. Then, for every $n \geq 2$, the quotient $G / K' \operatorname{Stab}_G(n)$ is a p -group of maximal class of order p^{n+1} .*

Proof. For simplicity, let us write $T_n = K' \operatorname{Stab}_G(n)$, $Q_n = G / T_n$ and $A_n = K / T_n$ (take into account that $\operatorname{Stab}_G(2) \leq G' \leq K$). Since $|Q_n : Q'_n| = |G : G'| = p^2$ and A_n is an abelian maximal subgroup of Q_n , it follows from Lemma 2.3 that Q_n is a p -group of maximal class. As a consequence, if we want to prove that $|Q_n| = p^{n+1}$, it suffices to see that the nilpotency class of Q_n is n .

We need an auxiliary result. Let $\{x_i\}_{i \geq 1}$ be a sequence of elements of G such that $\{x_1, x_2\} = \{a, b\}$ and $x_i \in \{a, b\}$ for every $i \geq 3$. We claim that, for every $i \geq 2$, the section $\gamma_i(Q_n) / \gamma_{i+1}(Q_n)$ is generated by the image of the commutator $[x_1, x_2, \dots, x_i]$. We argue by induction on i . If $i = 2$ then we have to show that the image of $[a, b]$ generates $\gamma_2(Q_n) / \gamma_3(Q_n)$. This follows immediately from (i) in Lemma 2.3, since $[a, b] = [a, a^{-1}b]$, where $bT_n \in Q_n \setminus A_n$ and $a^{-1}bT_n = (ba^{-1}T_n)^a \in A_n \setminus \gamma_2(Q_n)$. Now, if we assume that the result holds for $i - 1$, we get it for i by using (ii) of Lemma 2.3.

Let us now prove that the class of Q_n is n , by induction on n . Assume first that $n = 2$. We have

$$\psi([b, a]) = (a^{-1}b, 1, \dots, 1, b^{-1}a)$$

and

$$\psi([b, a, b]) = ([a^{-1}b, a], 1, \dots, 1, [b^{-1}a, b]) = ([b, a], 1, \dots, 1, [a, b]),$$

so that $[b, a, b] \in \text{Stab}_G(2)$. It follows that the image of $[b, a, b]$ in Q_2 is trivial. By the previous paragraph, we necessarily have $\gamma_3(Q_2) = \gamma_4(Q_2)$. Hence $\gamma_3(Q_2) = 1$, and the class of Q_2 is at most 2. If Q_2 is of class 1, then $[b, a] \in K' \text{Stab}_G(2)$ and, by Lemma 4.4, $a^{-1}b \in K' \text{Stab}_G(1)$. Hence $a^{-1} \in \text{Stab}_G(1)$, which is a contradiction. Thus Q_2 is of class 2.

Now we assume the result for $n - 1$, and we prove it for n . We have

$$\psi([b, a, b, \overset{n-1}{\cdot}, b]) = ([b, a, \overset{n-1}{\cdot}, a], 1, \dots, 1, [a, b, \overset{n-1}{\cdot}, b]),$$

and

$$[b, a, \overset{n-1}{\cdot}, a], [a, b, \overset{n-1}{\cdot}, b] \in K' \text{Stab}_G(n - 1),$$

since Q_{n-1} has class $n - 1$ by the induction hypothesis. Thus

$$(18) \quad \psi([b, a, b, \overset{n-1}{\cdot}, b]) \in K' \text{Stab}_G(n - 1) \times \overset{p}{\cdot} \times K' \text{Stab}_G(n - 1).$$

Now,

$$\begin{aligned} & (K' \text{Stab}_G(n - 1) \times \overset{p}{\cdot} \times K' \text{Stab}_G(n - 1)) \cap \psi(G) \\ &= (K' \times \overset{p}{\cdot} \times K')(\text{Stab}_G(n - 1) \times \overset{p}{\cdot} \times \text{Stab}_G(n - 1)) \cap \psi(G) \\ &\subseteq \psi(K')(\text{Stab}_G(n - 1) \times \overset{p}{\cdot} \times \text{Stab}_G(n - 1)) \cap \psi(G) \\ &= \psi(K')(\text{Stab}_G(n - 1) \times \overset{p}{\cdot} \times \text{Stab}_G(n - 1) \cap \psi(G)) \\ &= \psi(K')\psi(\text{Stab}_G(n)) = \psi(K' \text{Stab}_G(n)). \end{aligned}$$

It follows that $[b, a, b, \overset{n-1}{\cdot}, b] \in K' \text{Stab}_G(n)$, and so this commutator becomes trivial in Q_n . Since the image of this commutator generates the quotient $\gamma_{n+1}(Q_n)/\gamma_{n+2}(Q_n)$, we have $\gamma_{n+1}(Q_n) = 1$. Hence the class of Q_n is at most n .

If Q_n has class strictly less than n , then since the image of $[b, a, b, \overset{n-2}{\cdot}, b]$ generates $\gamma_n(Q_n)/\gamma_{n+1}(Q_n)$, it follows that

$$[b, a, b, \overset{n-2}{\cdot}, b] \in K' \text{Stab}_G(n).$$

Since

$$\psi([b, a, b, \overset{n-2}{\cdot}, b]) = ([b, a, \overset{n-2}{\cdot}, a], 1, \dots, 1, [a, b, \overset{n-2}{\cdot}, b]),$$

it follows from Lemma 4.4 that

$$[b, a, \overset{n-2}{\cdot}, a] \in K' \text{Stab}_G(n - 1).$$

This is a contradiction, since Q_{n-1} is of class $n - 1$, and $\gamma_{n-1}(Q_{n-1})/\gamma_n(Q_{n-1})$ is generated by the image of $[b, a, \overset{n-2}{\cdot}, a]$. Thus we conclude that the nilpotency class of Q_n is n , which completes the proof of the theorem. \square

Theorem 4.7. *Let G be a GGS-group with a constant defining vector. Then*

$$\log_p |G_n| = p^{n-1} + 1 - \frac{p^{n-2} - 1}{p - 1} - \frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

for every $n \geq 2$, and

$$\dim_{\Gamma} \overline{G} = \frac{p-2}{p-1}.$$

Proof. As on previous occasions, the formula for the Hausdorff dimension of \overline{G} is immediate once we obtain $\log_p |G_n|$. For that purpose, we argue by induction on n . If $n = 2$, then by Theorem 2.4, we have $\log_p |G_2| = t + 1$, where t is the rank of the matrix $C = C(1, \overset{p-1}{\cdot}, 1, 0)$. By Lemma 2.7, $p - t$ is the multiplicity of 1 as a root in \mathbb{F}_p of the polynomial $X^{p-2} + \dots + X + 1$. Thus $t = p$ and $\log_p |G_2| = p + 1$, as desired.

Assume now that $n \geq 3$. Let $K = \langle ba^{-1} \rangle^G$, and $L = \psi^{-1}(K' \times \overset{p}{\cdot} \times K')$. Then we have the following decomposition of the order of G_n :

$$(19) \quad |G_n| = |G : K' \text{Stab}_G(n)| |K' \text{Stab}_G(n) : L \text{Stab}_G(n)| |L \text{Stab}_G(n) : \text{Stab}_G(n)|.$$

By Theorem 4.6, we know that $|G : K' \text{Stab}_G(n)| = p^{n+1}$. On the other hand, since

$$K' \text{Stab}_G(n) / L \text{Stab}_G(n) \cong K / K' \text{Stab}_G(n-1) \times \overset{p-2}{\cdot} \times K / K' \text{Stab}_G(n-1)$$

by Theorem 4.5, and since $|K / K' \text{Stab}_G(n-1)| = p^{n-1}$ (again by Theorem 4.6), it follows that

$$|K' \text{Stab}_G(n) : L \text{Stab}_G(n)| = p^{(n-1)(p-2)}.$$

Finally,

$$\begin{aligned} |L \text{Stab}_G(n) : \text{Stab}_G(n)| &= |L : \text{Stab}_L(n)| = |\psi(L) : \psi(\text{Stab}_L(n))| \\ &= |K' \times \overset{p}{\cdot} \times K' : \text{Stab}_{K'}(n-1) \times \overset{p}{\cdot} \times \text{Stab}_{K'}(n-1)| \\ &= |K' : \text{Stab}_{K'}(n-1)|^p = |K' \text{Stab}_G(n-1) : \text{Stab}_G(n-1)|^p \\ &= |G / \text{Stab}_G(n-1)|^p / |G / K' \text{Stab}_G(n-1)|^p \\ &= |G_{n-1}|^p p^{-np}. \end{aligned}$$

Now, from (19) we get

$$\begin{aligned} \log_p |G_n| &= p \log_p |G_{n-1}| + n + 1 + (n-1)(p-2) - np \\ &= p \log_p |G_{n-1}| - n - p + 3, \end{aligned}$$

and the result follows by applying the induction hypothesis to G_{n-1} . \square

REFERENCES

- [1] A.G. Abercrombie, Subgroups and subrings of profinite rings, *Math. Proc. Camb. Phil. Soc.* **116** (1994), 209–222.
- [2] M. Abért, B. Virág, Dimension and randomness in groups acting on rooted trees, *J. Amer. Math. Soc.* **18** (2005), 157–192.
- [3] Y. Barnea, A. Shalev, Hausdorff dimension, pro- p groups, and Kac-Moody algebras, *Trans. Amer. Math. Soc.* **349** (1997), 5073–5091.
- [4] L. Bartholdi, R.I. Grigorchuk, On parabolic subgroups and Hecke algebras of some fractal groups, *Serdica Math. J.* **28** (2002), 47–90.

- [5] L. Bartholdi, R.I. Grigorchuk, Z. Šunić, Branch groups, in Handbook of Algebra, Vol. 3, 989–1112, North-Holland, 2003.
- [6] Y. Berkovich, Groups of Prime Power Order, Volume 1, de Gruyter, 2008.
- [7] D. Cox, Galois Theory, Wiley-Interscience, 2004.
- [8] J. Fabrykowski, N. Gupta, On groups with sub-exponential growth functions, *J. Indian Math. Soc.* **49** (1985), 249–256.
- [9] G.A. Fernández-Alcober, A. Zugadi-Reizabal, Spinal groups: semidirect product decompositions and Hausdorff dimension, *J. Group Theory* **14** (2011), 491–519.
- [10] R.I. Grigorchuk, On Burnside’s problem on periodic groups, *Functional Anal. Appl.* **14** (1980), 41–43.
- [11] N. Gupta, S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.
- [12] E. Pervova, Profinite topologies in just infinite branch groups, preprint 2002–154 of the Max Planck Institute for Mathematics, Bonn, Germany.
- [13] E. Pervova, Profinite completions of some groups acting on trees, *J. Algebra* **310** (2007), 858–879.
- [14] A.V. Rozhkov, Finiteness conditions in groups of tree automorphisms, Habilitation thesis, Chelyabinsk, 1996. (In Russian.)
- [15] S. Sidki, On a 2-generated infinite 3-group: subgroups and automorphisms, *J. Algebra* **110** (1987), 24–55.
- [16] O. Siegenthaler, Hausdorff dimension of some groups acting on the binary tree, *J. Group Theory* **11** (2008), 555–567.
- [17] Z. Šunić, Hausdorff dimension in a family of self-similar groups, *Geom. Dedicata* **124** (2007), 213–236.
- [18] T. Vovkivsky, Infinite torsion groups arising as generalizations of the second Grigorchuk group, Proceedings of the International Algebraic Conference on the Occasion of the 90th Birthday of A.G. Kurosh, 357–377, de Gruyter, 2000.

MATEMATIKA SAILA, EUSKAL HERRIKO UNIBERTSITATEA, 48080 BILBAO (SPAIN)
E-mail address: `gustavo.fernandez@ehu.es`

MATEMATIKA SAILA, EUSKAL HERRIKO UNIBERTSITATEA, 48080 BILBAO (SPAIN)
E-mail address: `amaia.zugadi@ehu.es`